

Generalized Offset Pythagorean Stretches in Box-Pleated Uniaxial Bases

Robert J. Lang and Mu-Tsun Tsai

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Abstract

In the origami technique of box-pleating, most creases lie on a square grid, with most creases running vertically, horizontally, or at odd multiples of 45° . In the mid-1990s, several artists began designing origami figures using the square grid and crease directions of box-pleating, but incorporating concepts from circle packing, notably the use of polygons (rather than circles) to represent flaps and the area that they take up. Although these concepts could be seen in many designs created through the 1990s and 2000s, author Lang codified many of these techniques in [Lang \(2011\)](#) under the general name *polygon packing* and presented a technique of allowing the packing polygons to overlap one another, thereby giving more efficient design solutions, at the cost of moving some of the creases off of the underlying grid. The construction was called a *Pythagorean stretch*, reflecting the significance of right triangles and the distance relationship of the Pythagorean Theorem in their construction.

For some configurations of a Pythagorean stretch, it is possible to get the vertices between ridge creases to fall on grid points; this occurs when key distances between point triples form Pythagorean triples. Some examples of this condition were given in [Lang \(2011\)](#), but there are many examples of such constructions in the earlier works of Satoshi Kamiya (see, e.g., [Kamiya \(2002, 2003\)](#)). Kamiya made particular use of the $(3, 4, 5)$ right triangle, and this arrangement is often referred to as the “Kamiya pattern” [Murakami \(2017\)](#).

In this work, we present a general construction for ridge folds in the case of overlapping rectilinear polygons that allows one to get a maximal number of vertices on grid points. We call this construction a *generalized offset Pythagorean stretch*, or GOPS. We give explicit conditions for its existence and when all-grid ridge vertices are possible. We then show that Kamiya patterns, Pythagorean stretches, and offset Pythagorean stretches can all be constructed by either single GOPS or combinations of GOPSs constructed by techniques we introduce as *relay* and *fusion*. We close with several examples of the technique’s application.

The configuration of an *offset Pythagorean stretch* (OPS), as illustrated in ([Lang, 2011](#), Figure 14.22, p. 645), is shown (with some modifications) in Figure [1\(a\)](#) and (b). We generalize the OPS as GOPS in Figure [1\(c\)](#).

A key geometric feature of our construction is the region of overlap of the packing polygons,

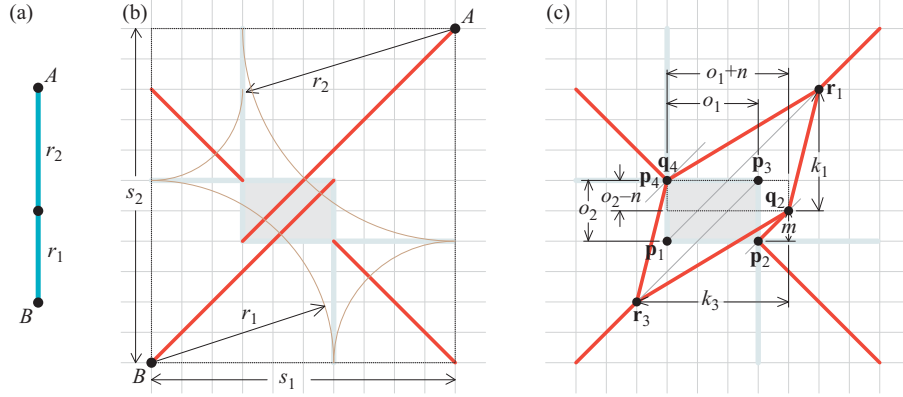


Figure 1: Schematic of two overlapping rectilinear polygons in box-pleating. (a) The tree graph for two flaps, A and B , with lengths r_1 and r_2 , respectively. (b) The crease pattern, with overlapping packing polygons for flaps A and B . Hinge lines are light blue; ridge creases, normally constructed as the straight skeleton of hinge lines, are red. (c) Parameterization of the vertices of a GOPS.

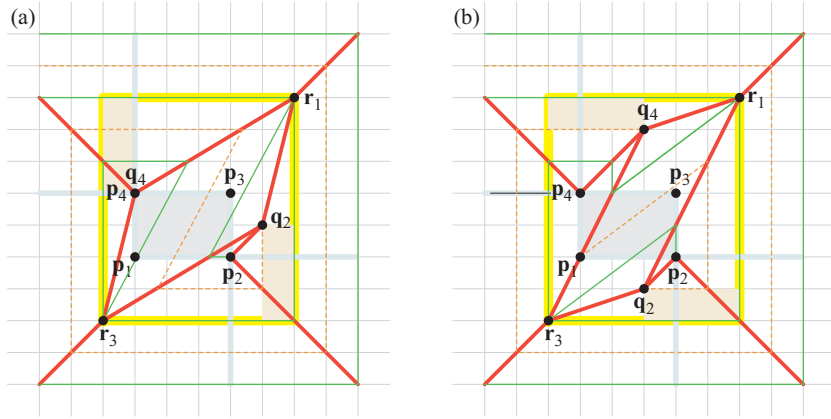


Figure 2: A conjugate pair solution for a overlap rectangle 3×2 with factorization 1×3 . Axial contours (solid green) and axial+1 contours (dashed amber) are shown. Left: The $(3,2):(3,1)$ GOPS. Right: the $(3,2):(1,3)$ GOPS.

shown in gray in Figure 1, with width o_1 and height o_2 . We show that there is a flat-foldable quadrilateral with all four vertices on grid points for that overlap rectangle if and only if there is a factorization

$$u \times v = o_1 o_2 / 2, \quad (1)$$

with both u and v integers. For each such factorization, there are two distinct integer-grid patterns of ridge creases, which we call a *conjugate pair*, labeling each solution by its corresponding values $(o_1, o_2):(u, v)$. One such pair is shown in Figure 2. Our construction gives all possible integer-grid patterns, in particular, Kamiya patterns with all possible Pythagorean triples.

We further show a similar analysis for the recently-described Generalized Pythagorean Stretch [Lang \(2017\)](#) and methods for combining GOPS and GPS in more complex patterns that are analogous to the Universal Molecule of circle-packing.